Lecture 3 More Geotherms

Heat Advection

Rocks sometimes move around at rates faster than heat conducts, so we need some way of accounting for this 'advection'. If the velocity of the rock is given by v, a new term is added to the heat conduction equation:

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T + v \bullet \nabla T$$

In 2D this is

$$\frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) + v \left(\frac{\partial T}{\partial x} + \frac{\partial T}{\partial z} \right)$$

A Half-Space Cooling Model for Oceanic Heat Flow and Temperature

As an oceanic plate ages, it cools, contracts, and subsides. For crust <70 Ma, the subsidence follows the square root of the age, and for crust <120 Ma, the heat flow follows the square root of the age; therefore we desire an expression showing $T(x,y,z,t)$ and $Q(x,y,z,t)$ proportional to $t^{1/2}$.

If the temperature field is in steady state, there is no heat production (a reasonable simplification for oceanic lithosphere), and there is no advection in the *z* direction (also true), the above equation can be simplified to

$$\kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) = -v \frac{\partial T}{\partial x}$$

If advection greatly outweighs horizontal heat conduction, we can neglect the curvature of the *T* field, $\partial^2 T/\partial x^2$, giving

$$\kappa \frac{\partial^2 T}{\partial z^2} = -v \frac{\partial T}{\partial x}$$

Recognizing that $v/x = 1/t$, this can be recast as our old friend the diffusion equation(!):

$$\kappa \frac{\partial^2 T}{\partial z^2} = -\frac{\partial T}{\partial t}$$

We did not discuss this earlier, but the solution of the diffusion equation is:

$$T = T_o erf\left(\frac{z}{2\sqrt{\kappa t}} \right)$$

Where *erf*, an *error function*, is defined by

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$$

The *error function* can be approximated analytically to better than 0.1% as

$$erf(x) = 1.008 - 2.2786x + 2.1166x^2 - 0.99597x^3 + 0.23588x^4 - 0.022371x^5$$

Approximate boundary conditions for a spreading ridge are $T = T_{asthenosphere}$ at $x = 0$ and $T = 0$ at $z = 0$. In other words, the solution to the diffusion equation, describing the temperature field as a function of age and depth is

$$T = T_{asthenosphere} erf\left(\frac{z}{2\sqrt{\kappa t}}\right)$$

The heat flow, obtained by differentiation, is then

$$Q = -k\left(\frac{\partial T}{\partial z}\right)_{z=0} = -\frac{kT_{asthenosphere}}{\sqrt{\pi \kappa t}}$$

Continental Heat Flow and Temperature

Recall the beautiful heat conduction equation

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_P}\left(\frac{\partial^2 T}{\partial z^2}\right) + \frac{A}{\rho C_P}$$

At steady state ($dT/dt = 0$) this simplifies to

$$\frac{\partial^2 T}{\partial z^2} = -\frac{A}{k}$$

In continents, a linear relationship exists between surface heat flow Q_S and near-surface radiogenic heat production A_S:

$$Q_S = Q_r + A_S z_A$$

where Q_r and z_A are constants for a given heat-flow province. For example,

	Q_S (mW/m^2)	Q_r (mW/m^2)	z_A (km)	A_S (μW/m^3)
Basin and Range	92	59	9.4	3.5
Western Australia	39	26	4.5	2.9

It is common to model heat generation in the continental crust as an exponentially decreasing function of depth down to some depth, z_{max}:

$$A(z) = A_S e^{-z/z_{\max}} \text{ for } z \le z_{\max}$$

Integrating the steady state heat conduction equation (above) gives

$$\frac{\partial T}{\partial z} = \frac{A_S}{k} z_A e^{-z/z_A} + C$$

We can use the boundary condition that at the surface ($z = 0$), the heat flow, $Q_S = k\partial T/\partial z$, to obtain

$$Q_S = k\left(\frac{A_S}{k} z_A + C\right) = A_S z_A + kC$$

and thus, C is

$$C = \frac{Q_S - A_S z_A}{k} = \frac{Q_r}{k}$$

Integrating a second time, and applying boundary conditions yields

$$T(z) = \frac{Q_r}{k} z + \frac{A_S}{k} {z_A}^2 \left(1 - e^{-z/z_A}\right)$$

which describes the temperature field in continental lithosphere as a function of reduced heat flow, Q_r, surface radiogenic heat production, A_S, depth of radiogenic layer, z_A, and depth, z.